

# GARSide STRUCTURE ON MONOIDS WITH QUADRATIC SQUARE-FREE RELATIONS

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**ABSTRACT.** We show the intimate connection between various mathematical notions that are currently under active investigation: a class of *Garside monoids*, with a “nice” Garside element, certain monoids  $S$  with quadratic relations, whose monoidal algebra  $A = \mathbf{k}S$  has a *Frobenius Koszul dual*  $A^!$  with *regular socle*, the monoids of *skew-polynomial type* (or equivalently, *binomial skew-polynomial rings*) which were introduced and studied by the author and in 1995 provided a new class of Noetherian *Artin-Schelter regular* domains, and *the square-free set-theoretic solutions of the Yang-Baxter equation*. There is a beautiful symmetry in these objects due to their nice combinatorial and algebraic properties.

## 1. INTRODUCTION

Let  $X$  be a nonempty set and let  $r : X \times X \longrightarrow X \times X$  be a bijective map. In this case we shall use notation  $(X, r)$  and refer to it as a *quadratic set* or set with quadratic map  $r$ . We present the image of  $(x, y)$  under  $r$  as

$$(1.1) \quad r(x, y) = ({}^x y, x^y).$$

The formula (1.1) defines a “left action”  $\mathcal{L} : X \times X \longrightarrow X$ , and a “right action”  $\mathcal{R} : X \times X \longrightarrow X$ , on  $X$  as:

$$(1.2) \quad \mathcal{L}_x(y) = {}^x y, \quad \mathcal{R}_y(x) = x^y,$$

for all  $x, y \in X$ . The map  $r$  is *left nondegenerate* (respectively, *right nondegenerate*), if the maps  $\mathcal{L}_x$  (respectively,  $\mathcal{R}_x$ ) are bijective for each  $x \in X$ . We say that  $r$  is *nondegenerate* if it is both left and right nondegenerate,  $r$  is *involution* if  $r^2 = id_{X \times X}$ . In this paper we shall always assume that  $r$  is nondegenerate. Also, as a notational tool, we shall often identify the sets  $X^{\times k}$  of ordered  $k$ -tuples,  $k \geq 2$ , and  $X^k$ , the set of all monomials of length  $k$  in the free monoid  $\langle X \rangle$ .

As in [16, 18, 19, 20, 21], to each quadratic set  $(X, r)$  we associate canonically algebraic objects (see Definition 1.1) generated by  $X$  and with quadratic defining relations  $\mathfrak{R} = \mathfrak{R}(r)$  naturally determined as

$$(1.3) \quad \begin{aligned} xy = y'x' \in \mathfrak{R}(r), \quad & \text{whenever} \\ r(x, y) = (y', x') \quad & \text{and} \quad (x, y) \neq (y', x') \text{ hold in } X \times X. \end{aligned}$$

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Note that in the case when  $X$  is finite, the set  $\mathfrak{R}(r)$  of defining relations is also finite, therefore the associated algebraic objects are finitely presented. We can tell the precise number of defining relations, whenever  $(X, r)$  is nondegenerate and involutive, with  $|X| = n$ . In this case the set of defining relations  $\mathfrak{R}(r)$ , contains precisely  $\binom{n}{2}$  quadratic relations (see Proposition 2.3). In many cases the associated algebraic objects will be *standard finitely presented* with respect to the degree-lexicographic ordering induced by an appropriate enumeration of  $X$ . It is known in particular that the algebra generated by the monoid  $S(X, r)$  defined in this way has remarkable homological properties when  $r$  is involutive, nondegenerate and obeys the braid or Yang-Baxter equation in  $X \times X \times X$ . Set-theoretic solutions were introduced in [6, 44] and have been under intensive study during the last decade. There are many works on set-theoretic solutions and related structures, of which a relevant selection for the interested reader is [44, 23, 34, 7, 29, 16, 15, 35, 37, 45, 18, 19, 20, 21, 5]. In this section we recall basic notions and results which will be used in the paper. We shall use the terminology, notation and some results from our previous works, [14, 15, 16, 19]. This paper is a natural continuation of [15] which provides results of significant importance for the Main Theorem.

Here we study the close relations between various notions that appear in the literature: square-free solutions of the Yang-Baxter equation; a class of Garside structures (we call them *regular Garside monoids*); skew polynomial monoids; semi-groups of I type; and a class of monoids  $S$  whose monoidal algebra  $A = kS$  has Frobenius Koszul dual  $A^!$  with special properties. All these are also closely related to Artin-Schelter regularity of certain quadratic algebras.

For a non-empty set  $X$ , as usual, we denote by  $\langle X \rangle$ , and  $_{gr}\langle X \rangle$ , respectively, the free monoid and the free group generated by  $X$  and by  $\mathbf{k}\langle X \rangle$ - the free associative  $\mathbf{k}$ -algebra generated by  $X$ , where  $\mathbf{k}$  is an arbitrary field. For a set  $F \subseteq \mathbf{k}\langle X \rangle$ ,  $(F)$  denotes the two sided ideal of  $\mathbf{k}\langle X \rangle$  generated by  $F$ .

**Definition 1.1.** [16, 19] Assume that  $r : X^2 \longrightarrow X^2$  is a bijective map.

(i) The monoid

$$S = S(X, r) = \langle X; \mathfrak{R}(r) \rangle,$$

with a set of generators  $X$  and a set of defining relations  $\mathfrak{R}(r)$ , is called *the monoid associated with  $(X, r)$* .

(ii) The group  $G = G(X, r)$  associated with  $(X, r)$  is defined as

$$G = G(X, r) = _{gr}\langle X; \mathfrak{R}(r) \rangle.$$

(iii) For arbitrary fixed field  $\mathbf{k}$ , the  $\mathbf{k}$ -algebra associated with  $(X, r)$  is defined as

$$(1.4) \quad \mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X; \mathfrak{R}(r) \rangle \simeq \mathbf{k}\langle X \rangle / (\mathfrak{R}_0),$$

where  $\mathfrak{R}_0 = \mathfrak{R}_0(r)$  is the set of *binomials* in  $\mathbf{k}\langle X \rangle$ :

$$(1.5) \quad \mathfrak{R}_0 = \{xy - y'x' \mid xy = y'x' \in \mathfrak{R}(r)\}.$$

Clearly  $\mathcal{A}$  is a quadratic algebra, generated by  $X$  and with defining relations  $\mathfrak{R}_0(r)$ . Furthermore,  $\mathcal{A}$  is isomorphic to the monoid algebra  $\mathbf{k}S(X, r)$ .

*Remark 1.2.* Conversely, as in [23, 15], each finitely presented monoid  $S = \langle X; \mathfrak{R} \rangle$ , where  $\mathfrak{R}$  is a set of quadratic binomial relations  $xy = y'x'$  such that each monomial  $xy \in X^2$  occurs at most once in  $\mathfrak{R}$  defines canonically a quadratic set  $(X, r)$ . Let  $r$

be the involutive bijective map determined via  $\mathfrak{R}$  as follows

$$r(x, y) = \begin{cases} (y', x'), & \text{if } xy = y'x' \in \mathfrak{R} \\ (x, y) & \text{else.} \end{cases}$$

Clearly,  $(X, r)$  is a quadratic set and  $S \simeq S(X, r)$ .

*Remark 1.3.* When we study the monoid  $S = S(X, r)$ , the group  $G = G(X, r)$ , or the algebra  $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) \simeq \mathbf{k}[S]$  associated with  $(X, r)$ , it is convenient to use the action of the infinite groups,  $\mathcal{D}_k(r)$ , generated by maps associated with the quadratic relations, as follows. We consider the bijective maps

$$r^{ii+1} : X^k \longrightarrow X^k, \quad 1 \leq i \leq k-1, \quad \text{where} \quad r^{ii+1} = Id_{X^{i-1}} \times r \times Id_{X^{k-i-1}}.$$

Note that these maps are elements of the symmetric group  $Sym(X^k)$ . Then the group  $\mathcal{D}_k(r)$  generated by  $r^{ii+1}$ ,  $1 \leq i \leq k-1$ , acts on  $X^k$ . If  $r$  is involutive, the bijective maps  $r^{ii+1}$  are involutive, as well, so in this case  $\mathcal{D}_k(r)$  is the infinite group

$$(1.6) \quad \mathcal{D}_k(r) = \langle r^{ii+1} \mid (r^{ii+1})^2 = e, \quad 1 \leq i \leq k-1 \rangle.$$

Note first that  $\mathcal{D}_k(r)$  is isomorphic to  $(C_2)^{* (k-1)}$ , the free product of  $k-1$  cyclic groups of order 2. In particular, for  $k=3$ ,  $\mathcal{D}_3(r)$  is simply the *the infinite dihedral group*.

Secondly, note that if  $\omega$  is a monomial of length  $k$  in  $\langle X \rangle$ , the set of all monomials  $\omega' \in \langle X \rangle$  such that  $\omega' = \omega$  as elements of  $S$ , coincide with the orbit  $\mathcal{O}_{\mathcal{D}_k(r)}(\omega)$  in  $X^k$ . Analogous statements are true for  $G$  and  $\mathcal{A}$ .

**Definition 1.4.** Let  $(X, r)$  be a quadratic set.

- (1)  $(X, r)$  is said to be *square-free* if  $r(x, x) = (x, x)$  for all  $x \in X$ .
- (2)  $(X, r)$  is called a *quantum binomial set* if it is nondegenerate, involutive and square-free.
- (3)  $(X, r)$  is a *set-theoretic solution of the Yang-Baxter equation* (YBE) if the braid relation

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

holds in  $X \times X \times X$ . In this case  $(X, r)$  is also called a *braided set*. If in addition  $r$  is involutive  $(X, r)$  is called a *symmetric set*.

In various cases in a nondegenerate  $(X, r)$  the left and the right actions are inverses, i.e.  $\mathcal{R}_x = \mathcal{L}_x^{-1}$  and  $\mathcal{L}_x = \mathcal{R}_x^{-1}$  for all  $x \in X$ . For example this is true for every square-free symmetric set  $(X, r)$ , see [14, 16]. In [19], we singled out a class of nondegenerate sets  $(X, r)$  by a condition **lri** defined below.

**Definition 1.5.** [19] Let  $(X, r)$  be a quadratic set (of arbitrary cardinality). We define the condition

$$(1.7) \quad \mathbf{lri}: \quad ({}^x y)^x = y = {}^x (y^x) \text{ for all } x, y \in X.$$

In other words **lri** holds if and only if  $\mathcal{R}_x = \mathcal{L}_x^{-1}$  and  $\mathcal{L}_x = \mathcal{R}_x^{-1}$ .

**Definition 1.6.** [27]

- (1) A *pre-Garside monoid* is a pair consisting of a monoid  $M$  and an element  $\Delta \in M$  such that the set  $\Sigma = \Sigma_\Delta$  of all left divisors of  $\Delta$  satisfies the following conditions:
  - (a)  $\Sigma$  is finite, generates  $M$ , and coincides with the set of right divisors of  $\Delta$ .

(b) If  $a, b \in \Sigma$  are such that  $\Delta a = \Delta b$ , or  $a\Delta = b\Delta$ , then  $a = b$

The element  $\Delta \in M$  is called *the Garside element of  $M$* .

- (2) Let  $(M, \Delta)$  be a pre-Garside monoid and let  $\Sigma$  be the set of left (and right) divisors of  $\Delta$ . The pair  $(M, \Delta)$  is a *Garside monoid* if  $M$  is atomic and for any two atoms  $s, t$  of  $M$ , the set

$$\{a \in \Sigma \mid s \preceq a, \text{ and } t \preceq a\}$$

has a (unique) minimal element  $\Delta_{s,t}$  (with respect to  $\preceq$ ).

Here  $\preceq$  is the partial order on  $M$  defined as  $s \preceq a$ , if  $s$  is a left divisor of  $a$  that is  $a = s\alpha$  for some  $\alpha \in M$ . It is known that each Garside monoid  $M$  embeds in its group of fraction  $G_M$ .  $G_M$  is called a *Garside group*.

- (3) A set  $\Sigma \subset M$  is *comprehensive* if  $1 \in \Sigma$  and  $M$  has a presentation by generators and relations such that all generators and relators belong to  $\Sigma$ .

A Garside monoid  $(M, \Delta)$  is *comprehensive* if the set  $\Sigma$  of the divisors of  $\Delta$  is comprehensive.

*Remark 1.7.* By definition  $S = \langle X; \mathfrak{R} \rangle$  has quadratic defining relations, thus, it has a *length balanced presentation* so by [27, Lemma 6.4],  $S(X, r)$  is atomic, and the set of its atoms coincides with  $X$ .

We introduce now the notions of *regular quantum monoids* and *regular Garside monoids*. In each case the name *regular* comes in a natural way—we show that the two notions are equivalent, and the monoids have a rich list of ‘good’ properties. Furthermore, in each case the semigroup algebra  $\mathbf{k}S$  is *Artin-Schelter regular ring*, see Corollary 1.15. Artin-Schelter regularity, see [2], is an important notion in noncommutative geometry.

**Definition 1.8.** Let  $(X, r)$  be a finite quantum binomial set with  $\mathbf{lri}$ , and let  $|X| = n$ . Let  $S = S(X, r)$ ,  $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$  be respectively the associated monoid, and the associated quadratic algebra. Let  $\mathcal{A}^!$  be the Koszul dual of  $\mathcal{A}$ , see Definition 4.1.

- (1) A monomial  $\omega \in S$  of length  $k$  is called a *square-free monomial*, if its orbit  $\mathcal{O}_{\mathcal{D}_k(r)}(\omega)$  in  $X^k$  does not contain words of the shape  $\omega' = axxb$ , where  $x \in X, a, b \in \langle X \rangle$ . In other words there is no equality  $\omega = axxb$  as elements of  $S$ .
- (2) Let  $\omega_0 = x_1 x_2 \cdots x_n \in X^n$  be a square free element of  $S$ , such that all  $x_i$  are pairwise distinct, so we enumerate  $X = \{x_1, \dots, x_n\}$  and fix the degree-lexicographic ordering  $<$  on  $\langle X \rangle$ , where  $x_1 < x_2 < \cdots < x_n$ . We say that  $\omega_0$  is a *regular element* of  $S$  if it is the minimal element, with respect to  $<$ , in the orbit  $\mathcal{O}_{\mathcal{D}_k(r)}(\omega_0)$ . We shall also say that each  $\omega \in \mathcal{O}_{\mathcal{D}_k(r)}(\omega_0)$  has a *regular presentation*  $\omega = \omega_0$ .
- (3)  $S$  is called a *regular quantum monoid* if
  - (i) The Koszul dual  $\mathcal{A}^!$  is Frobenius of dimension  $n$ , and
  - (ii) The *principal monomial*  $W$  of  $S$  has a *regular presentation*  $W = x_1 x_2 \cdots x_n$ .  
 ( By definition  $W$  spans the socle of  $\mathcal{A}^!$  and is the longest square-free element in  $S$ , see section 4).
- (4) A Garside monoid  $S$  is a *regular Garside monoid* if
  - (i)  $S$  has a Garside element  $\Delta \in S$  with a regular presentation  $\Delta = x_1 x_2 \cdots x_n$  and

(ii) every square-free monomial  $a \in S$  of length  $\leq n$  is a left (and a right) divisor of  $\Delta$ .

In this case  $\Delta$  is called a *regular Garside element* of  $S$ . Clearly,  $\Delta$  is unique.

*Remark 1.9.* Note that, in general, a regular element  $\omega \in S$  can have more than one regular presentations, see Example 1.10. We don't know whether it is possible that the monoid  $S(X, r)$  of an arbitrary quantum binomial set  $(X, r)$  with **lri**, may have two distinct regular elements  $\omega, \omega'$  (in the sense that  $\omega \neq \omega'$  as elements of  $S$ ).

**Example 1.10.** \*\*\*

**Definition 1.11.** [10] We say that the monoid  $S$  is a *monoid of skew-polynomial type*, (or shortly, a *skew-polynomial semigroup*) if it has a standard finite presentation as

$$S = \langle X; \mathfrak{R} \rangle,$$

where the set of generators  $X$  is ordered:  $x_1 < x_2 < \dots < x_n$ , and  $\mathfrak{R}$  is a set of  $\binom{n}{2}$  quadratic relations,

$$(1.8) \quad \mathfrak{R} = \{x_j x_i = x_{i'} x_{j'} \mid 1 \leq i < j \leq n, 1 \leq i' < j' \leq n\},$$

satisfying

- (i) each monomial  $xy \in X^2$ , with  $x \neq y$ , occurs in exactly one relation in  $\mathfrak{R}$  (a monomial of the type  $xx$  does not occur in any relation in  $\mathfrak{R}$ );
- (ii) if  $(x_j x_i = x_{i'} x_{j'}) \in \mathfrak{R}$ , with  $1 \leq i < j \leq n$ , then  $i' < j'$ , and  $j > i'$ . (this also imply  $i < j'$ , see [10])
- (iii) the overlaps  $x_k x_j x_i$  with  $k > j > i$ ,  $1 \leq i, j, k \leq n$  do not give rise to new relations in  $S$ , or equivalently, see [4], the set of polynomials  $\mathfrak{R}_0$  is a Gröbner basis of the ideal  $(\mathfrak{R}_0)$  with respect to the degree-lexicographic ordering of the free semigroup  $\langle X \rangle$ .

*Remark 1.12.* Clearly, the set of relations (1.8) defines canonically an involutive bijective map

$$\begin{aligned} r : X \times X &\longrightarrow X \times X, \\ (x_j, x_i) &\leftrightarrow (x_{i'}, x_{j'}), \quad 1 \leq i < j \leq n, \quad j > i' \quad 1 \leq i' < j' \leq n \\ (x, x) &\leftrightarrow (x, x), \quad \forall x \in X. \end{aligned}$$

and  $(X, r)$  is a quantum binomial quadratic set.

*Remark 1.13.* The theory of (noncommutative) Gröbner bases implies that each monomial  $u \in S_0$  has a uniquely determined *normal form*,  $\text{Nor}(u)$ , that is the minimal (w.r.t.  $<$ ) element in the orbit  $\mathcal{D}_m(u)$ , where  $|u| = m$ . It is known that the normal form can be found effectively by applying finite steps of reductions determined via the relations. When the relations are of skew-polynomial type, as in 1.8, the normal form  $\text{Nor}(u)$  is an *ordered monomial*,  $(x_{i_1} \dots x_{i_m})$ , with  $i_1 \leq i_2 \leq \dots \leq i_m$ , and (since  $\mathfrak{R}$  is a Gröbner basis) any ordered monomial in  $\langle X \rangle$  is in normal form (mod  $(\mathfrak{R})$ ). In other words, condition (iii) of Definition 1.11 may be rephrased by saying that (as sets) we can identify  $S_0$  with *the set of ordered monomials*

$$\mathcal{N}_0 = \{x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid \alpha_n \geq 0 \text{ for } 1 \leq i \leq n\}.$$

We recall that the skew polynomial monoids appeared first in the context of a class of quadratic algebras with a Poincaré-Birkhoff-Witt type  $\mathbf{k}$ -basis, namely *the binomial skew polynomial rings*. These rings were introduced and studied in [9, 10, 11, 23, 15, 28]. Laffaille calls them *quantum binomial algebras* and uses computer programme to show in [28], that for  $|X| \leq 6$ , the associated automorphism  $R$  is a solution of the Yang-Baxter equation. We prefer to keep the name “*binomial skew polynomial rings*” since we have been using this name for already 15 years. We have verified in two different ways that the binomial skew polynomial rings provide a new (at that time) class of Artin-Schelter regular rings of global dimension  $n$ , where  $n$  is the number of generators  $X$ . In fact, the challenge was to prove the Gorenstein condition. The very first proof (1994) which the author reported on the Artin’s seminar MIT, involved spectral sequences. A second proof (which is not yet well known) is combinatorial, it involves the Frobenius property of the Koszul dual algebra  $A^!$ , see [11]. This result is also published in [15], Section 3. The third proof which is now well-known, was given in [23], and uses the good homological and algebraic properties of semigroups of I type and their semigroup algebras. Furthermore, a close relation was found in [23] between the three notions of semigroups of skew-polynomial type, semigroups of I-type and a class of set-theoretic solutions of YBE, the square-free solutions. In 1996 the author conjectured that the three notions are equivalent, see [12] (and also [14], [13]). The verification took several years, but gradually we better understood the rich structure and beautiful symmetry in the associated algebraic objects. These are now used to verify their close relation with a class of Garside structures. The study of Garside monoids and Garside groups has been recently intensified, Garside structures are used to prove recognizability of various properties in groups and have significant importance for the study of braid groups. Recently Chouraqui showed that for every finite nondegenerate symmetric set  $(X, r)$ , the associated monoid  $S(X, r)$  is a Garside monoid, hence the group  $G(X, r)$  is a Garside group.

We give priority here to the natural equivalence of the notions of various algebraic objects each of which is related to a quadratic algebra, that is “*a quantum space*” in the sense of Manin, [39], Section 3. Their equivalence makes it possible to combine and explore the “good” properties of these objects.

**Main Theorem 1.14.** *Let  $(X, r)$  be a finite quantum binomial quadratic set with  $\mathbf{lri}$  (see (1.7)). Let  $S = S(X, r)$  be the associated monoid. The following conditions are equivalent*

- (1)  *$S$  is a regular Garside monoid.*
- (2)  *$S$  is a regular quantum monoid.*
- (3)  *$S$  is a monoid of skew-polynomial type, (with respect to some appropriate enumeration of  $X$ ), or equivalently,  $A(k, X, r)$  is a binomial skew-polynomial ring.*
- (4)  *$(X, r)$  is a set-theoretic solution of the YBE.*

*Each of these conditions implies that  $(S, \Delta)$  is a comprehensive Garside monoid with a Garside element  $\Delta$  and its group of fractions is isomorphic to the group  $G = G(X, r)$  (with the same generators and relations). In particular  $(G, \Delta)$ , is a Garside group.*

**Corollary 1.15.** *Let  $(X, r)$  be a finite quantum binomial quadratic set with  $\mathbf{lri}$ . Each of the conditions (1), (2), (3), (4) of the Main Theorem implies that the*

associated quadratic algebra  $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$  is an Artin-Schelter regular ring of global dimension  $n$ , where  $n$  is the cardinality of  $X$ . Furthermore,  $\mathcal{A}$  is Koszul and left and right Noetherian domain.

## 2. PRELIMINARIES ON QUADRATIC SETS $(X, r)$ AND THE ASSOCIATED OBJECTS

In this section  $(X, r)$  is a nondegenerate quadratic set of arbitrary cardinality. In the cases when  $X$  is finite this will be clearly indicated. When this is possible we shall avoid a fixed enumeration of  $X$ , to make most of the properties invariant on the enumeration (see for example the cyclic conditions).

*Remark 2.1.* Suppose  $(X, r)$  is a quadratic set, and let  ${}^x\bullet$ , and  $\bullet^x$  be the associated left and right actions. Then

- (1)  $r$  is involutive if and only if

$${}^xy(x^y) = x, \text{ and } ({}^yx)^{y^x} = x, \text{ for all } x, y \in X.$$

- (2)  $r$  is square-free if and only if

$${}^xx = x, \text{ and } x^x = x \text{ for all } x \in X.$$

- (3) If  $r$  is nondegenerate and square-free, then

$${}^xy = x \iff x^y = y \iff y = x \iff r(x, y) = (x, y).$$

It is also straightforward to write out the Yang-Baxter equation for  $r$  in terms of the actions. This is in [7] (see also [19]), but we recall it here in our notations for convenience.

*Remark 2.2.* Let  $(X, r)$  be given in the notations above. Then  $r$  obeys the YBE (or  $(X, r)$  is a braided set) iff the following conditions hold

$$\mathbf{l1} : \quad {}^x(yz) = {}^xy({}^xz), \quad \mathbf{r1} : \quad (x^y)^z = (x^yz)^{y^z},$$

$$\mathbf{lr3} : \quad ({}^xy)^{({}^xz)} = ({}^xz)^{({}^xy)},$$

for all  $x, y, z \in X$ .

Some of the conditions of the following proposition can be extracted from [19], and possibly from other sources, see for example [19, Lemma 2.12, Corollary 2.13]. However, we prefer to present here a compact easy proof.

**Proposition 2.3.** *Suppose  $(X, r)$  is a quadratic set of arbitrary cardinality, and  $r$  is nondegenerate and involutive. Then the following conditions hold.*

- (1)  *$S$  satisfies cancelation law on monomials of length 2, that is for every  $s, a, b \in X$  the following implications hold*

$$(2.1) \quad \begin{aligned} (a) \quad & sa = sb \text{ holds in } S \implies a = b \\ (b) \quad & as = bs \text{ holds in } S \implies a = b \end{aligned}$$

- (2) **the left Ore condition on  $X^2$ :** *for every pair  $s, t \in X$  there exists a unique pair  $a, b \in X$ , such that  $sa = tb$ . Furthermore,  $a = b$  iff  $s = t$*

- (3) **the right Ore condition on  $X^2$ :** *for every pair  $s, t \in X$  there exists a unique pair  $a, b \in X$ , such that  $as = bt$ . Furthermore,  $a = b$  iff  $s = t$ .*

- (4) For each  $x \in X$  there exists unique  $y \in X$  such that  $r(x, y) = (x, y)$ . Moreover, if  $X$  is finite, of order  $n$ , then the set of defining relations  $\mathfrak{R}(r)$  contains exactly  $\binom{n}{2}$  relations, and  $r$  has exactly  $n$  "fixed points," that is pairs  $(x, y)$  with  $r(x, y) = (x, y)$ .

*Proof.* We shall prove part (1). Suppose  $sa = sb$  holds in  $S$ . Then (by the involutiveness of  $r$ ) exactly two cases are possible

- (i)  $(s, a) = (s, b)$  holds in  $X \times X$ ,
- (ii)  $r(s, a) = (s, b)$  and  $r(s, b) = (s, a)$  hold in  $X \times X$ .

In the case (i) the equality  $a = b$  is straightforward. Assume (ii) holds. Then

$$\begin{aligned} (s, b) = r(s, a) = ({}^s a, {}^s a) &\implies {}^s a = s \\ (s, a) = r(s, b) = ({}^s b, {}^s b) &\implies {}^s b = s \end{aligned}$$

This yields

$${}^s a = s \quad \text{and} \quad {}^s b = s,$$

which, by the (left) non degeneracy of  $r$  implies  $a = b$ .

We shall prove next part (2) of the lemma. Suppose  $s, t \in X$ . Then by the nondegeneracy of  $r$  there exists unique  $a \in X$ , such that  ${}^s a = t$ . Denote  $s^a = b$ . Then clearly,  $r(s, a) = ({}^s a, {}^s a) = (t, b)$  implies

$$sa = tb \quad \text{holds in } S.$$

That this equality determines  $a$  and  $b$  uniquely, follows from the "2-cancellation law" (2.1). The same law implies that  $a = b$  iff  $s = t$ .

Using the right nondegeneracy of  $r$  and analogous argument one proves part (3) of the proposition.

We now prove part (4) of the proposition. Let  $x \in X$ . Note first that for all  $x, y \in X$  the following implication holds.

$$(2.2) \quad {}^x y = x \implies r(x, y) = (x, y)$$

Indeed, suppose  ${}^x y = x$ . Then there are equalities in  $X \times X$ :

$$r(x, y) = ({}^x y, x^y) = (x, x^y)$$

Therefore, there is an equality in  $S$

$$xy = xx^y,$$

which, by the cancellation law (on generators), see (2.1) implies  $x^y = y$ .

By the nondegeneracy of  $r$ , for each  $x$  there exists a  $y = y_x \in X$  such that  ${}^x y = x$ , and therefore, by (2.2),  $r(x, y) = (x, y)$ . We claim that  $y = y_x$  is unique with this property. Indeed, an equality  $r(x, z) = (x, z)$  implies  ${}^x z = x = {}^x y$ , which by the (left) nondegeneracy of  $r$  implies  $z = y$ .

Suppose now that  $|X| = n$ . Then there are exactly  $n$  "fixed" pairs  $(x, y_x)$  with the property  $r(x, y_x) = (x, y_x)$ . The remaining  $n(n-1)$  pairs in  $(X \times X)$  satisfy  $r(a, b) \neq (a, b)$ , which gives exactly one relation

$$ab = {}^a b \cdot a^b \in \mathfrak{R}(r).$$

Clearly, then the number of defining relations is exactly  $\binom{n}{2}$ . This proves part (4). The proposition has been verified.  $\square$

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We recall the notion of "cyclic conditions" in terms of the left and right actions



**Definition 2.4.** Let  $(X, r)$  be a quadratic set. We define the conditions

$$\begin{array}{ll} \mathbf{cl1} : & {}^yx = {}^yx \text{ for all } x, y \in X; & \mathbf{cr1} : & x^y = x^y, \text{ for all } x, y \in X; \\ \mathbf{cl2} : & {}^x{}^yx = {}^yx, \text{ for all } x, y \in X; & \mathbf{cr2} : & x^{y^x} = x^y \text{ for all } x, y \in X. \end{array}$$

We say  $(X, r)$  is *weak cyclic* if **cl1**, **cr1** hold and is *cyclic* if all four of the above hold.

One can also define left-cyclic as **cl1**, **cl2** and similarly right-cyclic.

**Proposition 2.5.** [19] *Let  $(X, r)$  be a quadratic set (not necessarily square-free or finite). Then any two of the following conditions imply the remaining third condition.*

- (1)  $(X, r)$  is involutive
- (2)  $(X, r)$  is nondegenerate and cyclic.
- (3) **lri** holds.

*Remark 2.6.* In Proposition 2.5 one can replace (2) by the weaker condition  
(2')  $(X, r)$  nondegenerate and **cl1** holds.

**Corollary 2.7.** *Every quantum binomial quadratic set  $(X, r)$  with **lri** satisfies all cyclic conditions, see Definition 2.4.*

Clearly, **lri** implies that whatever property is satisfied by the left action, an analogous property is valid for the right action and vice versa. In particular, this is valid for the left and right ‘cyclic conditions’. The cyclic conditions were discovered first in 1990, when the author studied binomial rings with skew polynomial relation. It is interesting to know that the proofs of the good algebraic and homological properties of these algebras and monoids use in explicit or implicit form the existence of the full cyclic condition, see Definition 2.4. This includes the properties of being Noetherian, Gorenstein, therefore Artin-Shelter regular, being of  $I$ -type and “producing” solutions of YBE, see [9, 10, 11, 16, 15, 23, 22, 25]. In 1996 the author was aware that each (finite) square-free solution  $(X, r)$  satisfies the (full) cyclic conditions. This was reported in various talks as one of the evidences for our conjecture that every square-free solution of finite order can be “generated” from a binomial ring with skew polynomial relation, see for example [13], and [14].

Compared with these works, in [19] we do not assume that  $X$  is finite, initially the only restriction on the map  $r$  we impose is “ $r$  is nondegenerate”. and study the implication of the cyclic conditions on the properties of the actions.

*Remark 2.8.* The result of these paper extend in a natural way (in direction of Garside structures) the results in [15], which are essential in various proofs of the new results. Note that in most statements here we assume  $(X, r)$  satisfies **lri**, while in [15] we have assumed *the weak cyclic conditions*, see Definition 2.4. However, these two conditions are equivalent for  $(X, r)$  nondegenerate and involutive, see Proposition 2.5 and Remark 2.6. Still **lri** seems easy and natural to formulate and convenient to use as it guarantees that the left actions  $\mathcal{L}_x, x \in X$  determine uniquely the right actions ( $\mathcal{R}_x = (\mathcal{L}_x)^{-1}$  and therefore  $r$ ). **lri** is always satisfied in the case when  $(X, r)$  is a square-free solution of YBE (of arbitrary cardinality).

The following can be extracted from [19, Theorem 2.35], where more equivalent conditions are stated.

**Theorem 2.9.** [19] *Suppose  $(X, r)$  is a quantum binomial quadratic set of arbitrary cardinality (i.e. non-degenerate, involutive and square-free). Then the following conditions are equivalent:*

- (1)  $(X, r)$  is a set-theoretic solution of the Yang-Baxter equation.
- (2)  $(X, r)$  satisfies **1l**.
- (3)  $(X, r)$  satisfies **r1**.
- (4)  $(X, r)$  satisfies **lr3**

*In this case  $(X, r)$  is cyclic and satisfies **lri**.*

Several other characterizations are already known in the case when  $X$  is finite, see [23], [16], [15] from where we extract the following.

**Facts 2.10.** *Suppose that  $(X, r)$  is a finite quantum binomial set,  $|X| = n$ , let  $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$  be the associated quadratic algebra over a field  $\mathbf{k}$ .*

*Then any of the conditions (1), ..., (4) of Theorem 2.9 is equivalent to any of following:*

- (1)  $S(X, r)$  is a semigroup of skew polynomial type (with respect to some appropriate enumeration of  $X$ ).
- (2)  $\mathcal{A}$  is a binomial skew polynomial ring (with respect to some appropriate enumeration of  $X$ ), that is an analogue of the Poincaré-Birkhoff-Witt theorem holds.
- (3)  $(X, r)$  satisfies **lri** and  $S(X, r)$  is a regular quantum monoid.
- (4)  $S(X, r)$  is a semigroup of  $I$ -type, (see [23] for the definition).

Referring to condition (1) of the theorem briefly as YBE, [23, Theorem 1.3] gives (4)  $\iff$  YBE, [23, Theorem 1.1] gives (2)  $\implies$  YBE, while the reverse implication (2)  $\Leftarrow$  YBE follows from [16, Theorem 2.26] (see also [15, Theorem B]). Finally, (3)  $\iff$  YBE is shown in [15, Theorem B].

### 3. REGULAR GARSIDE MONOIDS ARE MONOIDS OF SKEW POLYNOMIAL TYPE

In this section  $(X, r)$  will denote a finite quantum binomial quadratic set with **lri**,  $|X| = n$ . We shall assume that  $S = S(x, r)$  is a regular Garside monoid, see Definition 1.6.

The following lemma will be used in the sequel. It is a modification of [15, Lemma 2.19].

**Lemma 3.1.** *Suppose  $(X, r)$  is a finite quantum binomial set with **lri**. Let  $\mathcal{O} = \mathcal{O}_{\mathcal{D}_3}(\omega)$  be an arbitrary orbit of the action of  $\mathcal{D}_3$  on  $X^3$ , denote by  $\Delta_i$  the diagonal of  $X^{\times i}$ ,  $2 \leq i \leq 3$ . Then the following conditions hold.*

- (1)  $\mathcal{O} \cap \Delta_3 \neq \emptyset$  if and only if  $\mathcal{O} = \{xxx\}$ , for some  $x \in X$ .
- (2)  $\mathcal{O} \cap ((\Delta_2 \times X \cup X \times \Delta_2) \setminus \Delta_3) \neq \emptyset$  if and only if  $|\mathcal{O}| = 3$ .
- (3) *In each of the cases  $\omega = yyx$ , or  $\omega = yxx$ , where  $x, y \in X, x \neq y$ , the orbit  $\mathcal{O}_{\mathcal{D}_3}(\omega)$  contains exactly 3 elements. More precisely, the following implications hold*

$$\begin{aligned}
 (yx = x_1y_1) \in \mathfrak{R} &\implies (yx_1 = x_2y_1) \in \mathfrak{R}, \text{ and} \\
 &\mathcal{O}_{\mathcal{D}_3}(yyx) = \{yyx, yx_1y_1, x_2y_1y_1\}, \\
 (yx = x_1y_1) \in \mathfrak{R} &\implies (y_1x = x_1y_2) \in \mathfrak{R}, \text{ and} \\
 &\mathcal{O}_{\mathcal{D}_3}(yxx) = \{yxx, x_1y_1x, x_1x_1y_2\},
 \end{aligned}$$

where  $y, y_1, y_2$  respectively  $x, x_1, x_2$ , are not necessarily pairwise distinct.

Furthermore, suppose  $<$  is an ordering on  $X$  such that every relation in  $\mathfrak{R}$  is of the type  $yx = x'y'$ , where  $y > x$ ,  $x' < y'$ , and  $y > x'$ . Then the orbit  $\mathcal{O}_{\mathcal{D}}(y_1y_2y_3)$  with  $y_1 \prec y_2 \prec y_3$  does not contain elements of the form  $xyx$ , or  $xyy$ ,  $x \neq y \in X$ .

**Theorem 3.2.** *Let  $S = S(X, r)$  be a regular Garside monoid, let  $\Delta = x_1x_2\cdots x_n$  be a regular presentation of its Garside element. Then  $S$  is a monoid of skew-polynomial type with respect to the canonical enumeration  $X = \{x_1, \dots, x_n\}$ .*

We shall verify the theorem in two steps. **Step I.** We prove that every monomial  $x_ix_j$ ,  $1 \leq i < j \leq n$ , is normal.

**Step II.** We show that  $\mathfrak{R}$  is a Gröbner basis, so condition (iii) of Definition 1.11 is also in force.

All proofs are combinatorial and remind the proofs in [15]. However, due to differences in hypothesis we write them explicitly.

*Remark 3.3.* Note that by hypothesis the monomial  $\omega = x_1x_2\cdots x_n \in \langle X \rangle$  is in normal form and therefore each subword  $a$  of  $\omega$  is also in normal form, in particular each  $x_ix_{i+1}$ ,  $1 \leq i \leq n-1$ , is in normal form.

We shall use the terminology of [15].

**Definition 3.4.** [15] Let  $S$  be a monoid, let  $w \in S$ . We say that  $h \in X$  is a *head* of  $w$  if  $w$  can be presented (in  $S$ ) as

$$w = hw_1,$$

where  $w_1 \in \langle X \rangle$  is a monomial of length  $|w_1| = |w| - 1$ . Analogously,  $t \in X$  is a *tail* of  $w$  if

$$w = w't \quad (\text{in } S_0)$$

for some  $w' \in \langle X \rangle$ , with  $|w'| = |w| - 1$ .

**Notation 3.5.**  $H_w$  will denote the set of all heads of  $w$  in  $S$ , respectively  $T_w$  will denote the set of all tails of  $w$ .

**Lemma 3.6.** *The Garside element  $\Delta$  satisfies the conditions:*

- (1)  $\Delta$  is a monomial of length  $n$ . There exist  $n!$  distinct words  $\omega_i \in \langle X \rangle$ ,  $1 \leq i \leq n!$ , for which the equalities  $\omega_i = \Delta$  hold in  $S$ . (These are the elements of the orbit  $\mathcal{O}_{\mathcal{D}_n}(x_1\cdots x_n)$ . We call them presentations of  $\Delta$ .)
- (2) Every  $x \in X$  is a left and right divisor of  $\Delta$ , so it occurs as a “head” (respectively, as a “tail”) of some presentation of  $\Delta$ . Thus there are equalities in  $S$ :

$$\begin{aligned} \Delta &= x_1w'_1 = x_2w'_2 = \cdots x_nw'_n \\ &= \omega_1x_1 = \omega_2x_2 = \cdots \omega_nx_n. \end{aligned}$$

- (3) Every square-free monomial  $a \in S$  of length  $k$ ,  $k \leq n$  has exactly  $k$  distinct “heads”,  $h_1, \dots, h_k$ , and exactly  $k$  distinct “tails”,  $t_1, \dots, t_k$ . The pair of sets  $H_a, T_a$  uniquely determine  $a$  in  $S$ .
- (4) Furthermore for  $1 \leq j < j+k \leq n$  the monomials  $\tau_{jj+k} = x_jx_{j+1}\cdots x_{j+k}$  satisfy

$$H_{\tau_{jj+k}} = T_{\tau_{jj+k}} = \{x_j, x_{j+1}, \dots, x_{j+k}\}$$

- (5)  $\Delta$  is the shortest monomial which “encodes” all the information about the relations  $\mathfrak{R}$ . More precisely, for any relation  $(xy = y'x') \in \mathfrak{R}$ , there exists an  $a \in \langle X \rangle$ , such that  $W_1 = xya$  and  $W_2 = y'x'a$  are (different) presentations of  $W$ .

**Lemma 3.7.** *For each integer  $j$ ,  $1 \leq j \leq n-1$ , let  $\xi_{j,j+1}, \dots, \xi_{j,n}$ ,  $\eta_{j,j+1}, \dots, \eta_{j,n}$  be the elements of  $X$  uniquely determined by the relations*

$$(3.1) \quad \begin{aligned} &(\xi_{j,j+1}\eta_{j,j+1} = x_jx_{j+1}) \in \mathfrak{R} \\ &(\xi_{j,j+2}\eta_{j,j+2} = \eta_{j,j+1}x_{j+2}) \in \mathfrak{R} \\ &\dots\dots\dots \\ &(\xi_{j,n-1}\eta_{j,n-1} = \eta_{j,n-2}x_{n-1}) \in \mathfrak{R} \\ &(\xi_{j,n}\eta_{j,n} = \eta_{j,n-1}x_n) \in \mathfrak{R}. \end{aligned}$$

Then for each  $j$ ,  $1 \leq j \leq n-1$ , the following conditions are in force:

- (1)  $\xi_{j,j+s} \neq \eta_{j,j+s-1}$ , for all  $s$ ,  $2 \leq s \leq n-j$ .
- (2) The equality

$$\xi_{j,j+1}\xi_{j,j+2}\dots\xi_{j,n} = x_{j+1}\dots x_n \quad \text{holds in } S.$$

- (3)  $x_{j+1}x_{j+2}\dots x_n\eta_{j,n} = x_jx_{j+1}\dots x_n$  holds in  $S$ .
- (4) The elements  $\eta_{j,n}, \eta_{j+1,n}, \dots, \eta_{n-1,n}$  are pairwise distinct.

*Proof.* Condition (1) follows from the Ore conditions on generators, see (2) and (3) of Proposition 2.3. To prove the remaining conditions we use decreasing induction on  $j$ ,  $1 \leq j \leq n-1$ .

**Step 1.**  $j = n-1$ . Clearly,  $x_{n-1}x_n$  is in normal form, so the relation in  $\mathfrak{R}$  in which it occurs has the shape  $x_{n-1}x_n = \xi_{n-1,n}\eta_{n-1,n}$ , with  $\xi_{n-1,n} > x_{n-1}$ . It follows then that  $\xi_{n-1,n} = x_n$  and  $x_{n-1}x_n = x_n\eta_{n-1,n}$ . This gives (2) and (3). There is nothing to prove in (4).

**Step 2.** We first prove (4) for all  $j$ ,  $1 \leq j < n-1$ . Assume that for all  $k$ ,  $n-1 \geq k > j$ , the elements  $x_k, x_{k+1}, \dots, x_n, \xi_{k,k+1}, \dots, \xi_{k,n}, \eta_{k,k+1}, \dots, \eta_{k,n}$  satisfy

$$(3.2) \quad \begin{aligned} &\xi_{k,k+1}\eta_{k,k+1} = x_kx_{k+1} \in \mathfrak{R} \\ &\xi_{k,k+2}\eta_{k,k+2} = \eta_{k,k+1}x_{k+2} \in \mathfrak{R} \\ &\dots\dots\dots \\ &\xi_{k,n-1}\eta_{k,n-1} = \eta_{k,n-2}x_{n-1} \in \mathfrak{R} \\ &\xi_{k,n}\eta_{k,n} = \eta_{k,n-1}x_n \in \mathfrak{R}, \end{aligned}$$

all  $\eta_{j+1,n}, \eta_{j+2,n}, \dots, \eta_{n-1,n}$  are pairwise distinct, and the modified conditions (4), in which “ $j$ ” is replaced by “ $k$ ” hold. Let  $\xi_{j,j+1}, \dots, \xi_{j,n}, \eta_{j,j+1}, \dots, \eta_{j,n}$  satisfy (3.2). We shall prove that  $\eta_{j,n} \neq \eta_{k,n}$ , for all  $k, j < k \neq n-1$ . Assume the contrary,

$$\eta_{j,n} = \eta_{k,n}$$

for some  $k > j$ . Consider the relations

$$(3.3) \quad \xi_{j,n}\eta_{j,n} = \eta_{j,n-1}x_n, \quad \xi_{k,n}\eta_{k,n} = \eta_{k,n-1}x_n.$$

By Proposition 2.3 the Ore condition holds, so (3.3) imply

$$\eta_{j,n-1} = \eta_{k,n-1}.$$

Using the same argument in  $n-k$  steps we obtain the equalities

$$\eta_{j,n} = \eta_{k,n}, \eta_{j,n-1} = \eta_{k,n-1}, \dots, \eta_{j,k+1} = \eta_{k,k+1}.$$

Now the relations

$$\xi_{j,k+1}\eta_{j,k+1} = \eta_{j,k}x_{k+1}, \quad \xi_{k,k+1}\eta_{k,k+1} = x_kx_{k+1},$$

and the Ore condition imply  $\eta_{j,k} = x_k$ . Thus, by (3.2) and (3.1) we obtain a relation

$$\xi_{j,k}x_k = \xi_{j,k-1}x_k \in \mathfrak{R}.$$

This is impossible, by Proposition 2.3. We have shown that the assumption  $\eta_{j,n} = \eta_{k,n}$ , for some  $k > j$ , leads to a contradiction. This proves (4) for all  $j, 1 \leq j \leq n-1$ .

We set

$$(3.4) \quad \eta_1 = \eta_{1,n}, \eta_2 = \eta_{2,n}, \dots, \eta_{n-1} = \eta_{n-1,n}.$$

Next we prove (2) and (3).

By the inductive assumption, for  $k > j$ , we have

$$\xi_{k,k+1}\xi_{k,k+2} \cdots \xi_{k,n} = x_{k+1} \cdots x_n,$$

and

$$x_{k+1} \cdots x_n \cdot \eta_{k+1} = x_k \cdots x_n.$$

Applying the relations (3.2) one easily sees, that

$$\xi_{j,j+1}\xi_{j,j+2} \cdots \xi_{j,n} \cdot \eta_{j,n} = x_j x_{j+1} \cdots x_n.$$

Denote

$$\omega_j = \xi_{j,j+1}\xi_{j,j+2} \cdots \xi_{j,n}.$$

We have to show that the normal form,  $Nor(\omega_j)$ , of  $\omega_j$  satisfies the equality of words in  $\langle X \rangle$

$$Nor(\omega_j) = x_{j+1}x_{j+2} \cdots x_n.$$

As a subword of length  $n-j$  of the presentation  $\Delta = x_1x_2 \cdots x_{j-1}w_j\eta_{j,n}$ , the monomial  $\omega_j$  has exactly  $n-j$  heads

$$(3.5) \quad h_1 < h_2 < \cdots < h_{n-j}.$$

But  $Nor(\omega_j) = \omega_j$ , is an equality in  $\mathcal{S}$ , so the monomial  $Nor(\omega_j)$  has the same heads as  $\omega_j$ . Furthermore, there is an equality of words in  $\langle X \rangle$ ,  $Nor(\omega_j) = h_1\omega'$ , where  $\omega'$  is a monomial of length  $n-j-1$ . First we see that  $h_1 \geq x_j$ . This follows immediately from the properties of the normal monomials and the relations

$$(3.6) \quad Nor(\omega_j)\eta_j = \omega_j\eta_j = x_jx_{j+1} \cdots x_n \in N.$$

Next we claim that  $h_1 > x_j$ . Assume the contrary,  $h_1 = x_j$ . Then by (3.6) one has

$$x_j\omega'\eta_j = \omega_j\eta_j = x_jx_{j+1} \cdots x_n.$$

As a Garside monoid,  $S$  is cancellative, hence

$$\omega'\eta_j = x_{j+1} \cdots x_n \in N.$$

Thus  $\eta_j$  is a tail of the monomial  $x_{j+1} \cdots x_n$ . By the inductive assumption, conditions (2) and (3) are satisfied, which together with (3.4) give additional  $n-j$  distinct tails of the monomial  $x_{j+1} \cdots x_n$ , namely  $\eta_{j+1}, \eta_{j+2}, \dots, \eta_{n-1}, x_n$ . It follows then that the monomial  $x_{j+1} \cdots x_n$  of length  $n-j$  has  $n-j+1$  distinct tails, which is impossible. This implies  $h_1 > x_j$ . Now since  $\omega_j$  has precisely  $n-j = |\omega_j|$  distinct heads, which in addition satisfy (3.5) we obtain equality of sets

$$H_{\omega_j} = \{h_1, h_2, \dots, h_{n-j}\} = \{x_{j+1}, x_{j+2}, \dots, x_n\}.$$

Recall that by the inductive assumption we have

$$(3.7) \quad H_{x_{j+1}x_{j+2} \cdots x_n} = \{x_{j+1}, x_{j+2}, \dots, x_n\}.$$

The equality  $H_{\omega_j} = H_{x_{j+1}x_{j+2}\cdots x_n}$  clearly implies equality of monomials in  $S$  :

$$\omega_j = x_{j+1}x_{j+2}\cdots x_n.$$

We have shown (3). Now the equalities

$$x_{j+1}\cdots x_n\eta_j = x_jx_{j+1}\cdots x_n$$

and (3.7) yield :

$$H_{x_jx_{j+1}\cdots x_n} = \{x_j, x_{j+1}, x_{j+2}, \cdots, x_n\},$$

which proves (2). The lemma has been proved.  $\square$

**Proposition 3.8.** *In notation as in Lemma 3.7 the following conditions hold in  $S$ .*

- (1) *For any integer  $j$ ,  $1 \leq j \leq n-1$ , there exists unique  $\eta_j \in X$ , such that*

$$x_{j+1}\cdots x_n\eta_j = x_jx_{j+1}\cdots x_n.$$

- (2) *The elements  $\eta_1, \eta_2, \cdots, \eta_{n-1}$  are pairwise distinct.*

- (3) *For each  $j$ ,  $1 \leq j \leq n-1$ , the set of heads  $H_{W_j}$  of the monomial  $W_j = x_jx_{j+1}\cdots x_n$  is*

$$H_{W_j} = \{x_j, x_{j+1}, \cdots, x_n\}.$$

- (4) *For any pair of integers  $i, j$ ,  $1 \leq i < j \leq n$ , the monomial  $x_i x_j$  is normal. Furthermore, the unique relation in which  $x_i x_j$  occurs has the form  $x_j x_{j'} = x_i x_j$ , with  $j' > i'$ , and  $j' > i$ .*

*Proof.* Conditions (1), (2), (3) of the proposition follow from Lemma 3.7, here  $\eta_j$  are defined in (3.4). We shall prove first that for any pair  $i, j$ ,  $1 \leq i < j \leq n$ , the monomial  $x_i x_j$  is normal. Assume the contrary. Then there is a relation

$$(3.8) \quad (x_i x_j = x_j x_{j'}) \in \mathfrak{R}, \quad \text{where } j' < i.$$

Clearly,

$$(3.9) \quad (x_j x_{j+1} \cdots x_n) \eta_{j-1} \eta_{j-2} \cdots \eta_{i+1} = x_{i+1} x_{i+2} \cdots x_n,$$

holds in  $S$ . Consider the monomial  $u = x_i x_j x_{j+1} \cdots x_n \eta_{j-1} \eta_{j-2} \cdots \eta_{i+1} \in X^{n-i+1}$ . It satisfies the following equalities in  $S$

$$\begin{aligned} u &= x_i x_j x_{j+1} x_{j+2} \cdots x_n \eta_{j-1} \eta_{j-2} \cdots \eta_{i+1} \\ &= x_{i+1} x_{i+2} \cdots x_n \quad \text{by (3.9)} \\ &= x_{j'} x_{j'+1} \cdots x_n \eta_{j-1} \eta_{j-2} \cdots \eta_{i+1} \quad \text{by (3.8)} \end{aligned}$$

Note that the monomial  $x_{i+1} x_{i+2} \cdots x_n$  is in normal form, therefore  $\text{Nor } u = x_{i+1} x_{i+2} \cdots x_n$ . The inequality  $j' < i$  implies the obvious inequality in  $\langle X \rangle$  :

$$x_{j'} x_{j'+1} \cdots x_n \eta_{j-1} \eta_{j-2} \cdots \eta_{i+1} < x_{i+1} x_{i+2} \cdots x_n = \text{Nor } u,$$

which is impossible, since  $\text{Nor } u$  is the minimal element in the orbit  $\mathcal{D}_{n-i+1}(u)$ . We have proved that all monomials  $x_i x_j$  with  $1 \leq i < j \leq n$  are in normal form. Note that the number of relations is exactly  $\frac{n(n-1)}{2}$  and each relation contains exactly one normal monomial, it follows then that each monomial  $x_j x_i$ , with  $n \geq j > i \geq 1$ , is not in normal form. Hence, each relation in  $\mathfrak{R}$  has the shape  $y_j y_i = y_{j'} y_{i'}$ , where  $1 \leq i < j \leq n$ ,  $1 \leq i' < j' \leq n$ , and  $j > i'$ , which proves (3) and (4).  $\square$

Step I is complete now. This straightforwardly implies that the (unique) relation in  $\mathfrak{R}$  in which  $x_i x_j$  occurs has the shape

$$x_j x_{i'} = x_i x_j, \quad \text{with } j' > i', j' > i,$$

so the relations  $\mathfrak{R}$  have the “correct shape” of *skew-polynomial type*, thus conditions (1.8) (i) and (ii) of Definition 1.11 hold.

We next proceed with **Step II**.

**Lemma 3.9.** (a) *The set of relations  $\mathfrak{R}$  is a Gröbner basis with respect to the ordering  $<$  on  $\langle X \rangle$ .*

(b)  *$S(X, r) = \langle x_1 x_2 \cdots x_n \mid \mathfrak{R} \rangle$  is a semigroup of skew polynomial type.*

*Proof.* It will be enough to prove that the ambiguities  $y_k y_j y_i$ , with  $k > j > i$ , do not give rise to new relations in  $S$ . Or equivalently, each ordered monomial  $\omega$  of length 3, that is  $\omega = xyz, x \leq y \leq z$ , is in normal form.

Let  $\omega$  be an ordered monomial of length 3, and assume  $Nor_\omega \neq \omega$  (as words in  $\langle X \rangle$ ). This means there exists an  $\omega' \in \mathcal{D}_3(\omega)$ , such that  $\omega' < \omega$  (in the degree-lexicographic ordering  $<$  in  $\langle X \rangle$ ). Four cases are possible:

$$(3.10) \quad \begin{aligned} (i) \quad & \omega = x_i x_j x_k, 1 \leq i < j < k \leq n \\ (ii) \quad & \omega = x_i x_i x_j, 1 \leq i < j \leq n \\ (iii) \quad & \omega = x_i x_j x_j, 1 \leq i < j \leq n \\ (iv) \quad & \omega = x_i x_i x_i, 1 \leq i \leq n. \end{aligned}$$

Suppose (3.10) (i) holds. Then there is an equality of elements in  $S$ :

$$\omega = x_i x_j x_k = x_{i'} x_{j'} x_{k'} = \omega', \quad \text{where } x_{i'} \leq x_{j'} \leq x_{k'},$$

and as elements of  $\langle X \rangle$ , the two monomials satisfy

$$(3.11) \quad \omega' = x_{i'} x_{j'} x_{k'} < x_i x_j x_k = \omega.$$

By (3.11) one has

$$(3.12) \quad x_{i'} \leq x_i.$$

We claim that there is an inequality  $x_{i'} < x_i$ . Indeed, it follows from Lemma 3.1 that the orbit  $\mathcal{O}_{\mathcal{D}_3}(y_i y_j y_k)$  does not contain elements of the shape  $xyy$ , or  $xyy$ , therefore an assumption,  $x_i = x_{i'}$  would imply  $x_j x_k = x_{j'} x_{k'}$  with  $x_j < x_k$  and  $x_{j'} < x_{k'}$ , which contradicts Proposition 3.8. We have obtained that  $x_{i'} < x_i$ . One can easily see that there exists an  $\omega \in \langle X \rangle$ , such that

$$(x_i x_j x_k) * \omega = x_i x_{i+1} \cdots x_n.$$

The monomial  $x_i x_{i+1} \cdots x_n$  is in normal form, hence

$$Nor((x_i x_j x_k) * \omega) = x_i x_{i+1} \cdots x_n$$

. But  $x_{i'} x_{j'} x_{k'} = x_i x_j x_k$  in  $S$ , hence

$$Nor(x_{i'} x_{j'} x_{k'} * \omega) = Nor(x_i x_j x_k * \omega) = x_i x_{i+1} \cdots x_n.$$

This together with the following inequalities in  $\langle X \rangle$ :

$$Nor(x_{i'} x_{j'} y_{k'} * \omega) \leq x_{i'} x_{j'} x_{k'} * \omega < x_i x_{i+1} \cdots x_n$$

give a contradiction. It follows then that monomial  $x_i x_j x_k$ ,  $i < j < k$ , is in normal form.

Suppose now (ii)  $\omega = x_i x_i x_j, 1 \leq i < j \leq n$ . It follows from Lemma 3.1 that the orbit  $\mathcal{O} = \mathcal{O}_{\mathcal{D}_3}(x_i x_i x_j)$  is the set

$$\mathcal{O} = \{\omega = x_i x_i x_j, \omega_2 = x_i x_{j'} x_{i'}, \omega_3 = x_{j''} x_{i'} x_{i'}\}$$

where

$$x_{j'} x_{i'} = x_i x_j \in \mathfrak{R}, \text{ with } x_i < x_{j'} > x_{i'}$$

and

$$x_{j''} x_{i'} = x_i x_{j'} \in \mathfrak{R}, \text{ with } x_{j''} > x_{i'}.$$

It is clear then that there are strict inequalities in  $\langle X \rangle$ :

$$\omega = x_i x_i x_j < \omega_2 = x_i x_{j'} x_{i'} < \omega_3 = x_{j''} x_{i'} x_{i'}$$

which gives  $Nor(\omega) = \omega$ .

The case (iii) is analogous to (ii). Case (iv) is straightforward, since all relations are square free. We have proved that all ordered monomials of length 3 are in normal form, and therefore the ambiguities  $x_k x_j x_i$  do not give rise to new relations in  $S$  (i.e. are solvable). It follows then that the set of relations  $\mathfrak{R}$  is a Gröbner basis.

We have verified all conditions in Definition 1.11, therefore  $S(X, r)$  is a monomial of skew polynomial type. The lemma has been proved.  $\square$

**Proof of Theorem 3.1.** Now Theorem 3.2 follows straightforwardly from Proposition 3.8 and Lemma 3.9.

#### 4. PROOF OF THE MAIN THEOREM

The Koszul dual  $A^!$  of a quadratic algebra  $A$  was introduced by Yu. Manin, see [39], 5.1. The properties of the two algebras  $A$  and  $A^!$  are closely related.

Suppose  $(X, r)$  is a finite quantum binomial quadratic set, for convenience we enumerate  $X = \{x_1, \dots, x_n\}$ . (In most of the cases the enumeration will be induced by a regular monomial of length  $n$ ). Suppose  $\mathfrak{R}_0 = \mathfrak{R}_0(r)$ , see (1.5) is the set of defining relations of the quantum binomial algebra  $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ , so  $\mathcal{A} = \mathbf{k}\langle X \rangle / (\mathfrak{R}_0)$ . Clearly,  $\mathcal{A}$  is a quadratic algebra, and one can extract from [39] an explicit presentation of its Koszul dual,  $\mathcal{A}^!$ .

**Definition 4.1.** The Koszul dual  $\mathcal{A}^!$  of  $\mathcal{A}$ , is the quadratic algebra,

$$\mathcal{A}^! := \mathbf{k}\langle \xi_1, \dots, \xi_n \rangle / (\mathfrak{R}_0^\perp),$$

where the set  $\mathfrak{R}_0^\perp$  contains precisely  $\binom{n}{2} + n$  relations of the following two types:

a) *binomials*:

$$\xi_j \xi_i + \xi_{i'} \xi_{j'} \in \mathfrak{R}_0^\perp, \text{ whenever } x_j x_i - x_{i'} x_{j'} \in \mathfrak{R}_0, 1 \leq i \neq j \leq n;$$

and b) *monomials*:

$$(\xi_i)^2 \in \mathfrak{R}_0^\perp, 1 \leq i \leq n.$$

*Remark 4.2.* [39] Consider the vector spaces  $V = \text{Span}_{\mathbf{k}}(x_1, x_2, \dots, x_n)$ ,  $V^* = \text{Span}_{\mathbf{k}}(\xi_1, \xi_2, \dots, \xi_n)$ , and define a bilinear pairing

$$\langle | \rangle : V^* \otimes V \longrightarrow \mathbf{k} \quad \text{by} \quad \langle \xi_i | x_j \rangle := \delta_{ij}.$$

Then the relations  $\mathfrak{R}_0^\perp$  generate a subspace in  $V^* \otimes V^*$  which is orthogonal to the subspace of  $V \otimes V$  generated by  $\mathfrak{R}_0$ .



**Definition 4.3.** We introduce a monoid  $S^! = S^!(X, r)$  with *zero element* denoted by 0.  $S^!$  is generated by  $X$  and has a presentation as

$$S^! := \langle X \mid \mathfrak{R}^! \rangle$$

where the set  $\mathfrak{R}^!$  contains  $\binom{n}{2} + n$  defining relations

$$\mathfrak{R}^! = \mathfrak{R}^!(r) := \mathfrak{R}(r) \bigcup \{xx = 0 \mid x \in X\}$$

Clearly, there is a canonical epimorphism  $\varphi : S(X, r) \longrightarrow S^!(X, r)$ ,

$$\varphi(u) = \begin{cases} u, & \text{if } u \in S \text{ is a square-free monomial} \\ 0 & \text{else.} \end{cases}$$

**Proposition 4.4.** [15] *Suppose we can fix an enumeration  $X = \{x_1, \dots, x_n\}$ , such that each relation in  $\mathfrak{R}_0$  has the shape*

$$x_j x_i - x_{i'} x_{j'}, \quad 1 \leq i < j \leq n, \quad 1 \leq i' < j' < n, \quad j > i'.$$

*As usual, we shall consider Gröbner bases with respect to degree-lexicographic ordering on  $\langle X \rangle$ . The following are equivalent.*

- (1) *The set  $\mathfrak{R}_0$  is a Gröbner basis of the ideal  $(\mathfrak{R}_0)$ , or equivalently, the algebra  $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$  is a binomial skew polynomial ring.*
- (2)  *$S(X, r)$  is a monoid of skew polynomial type.*
- (3) *The set of ordered monomials*

$$\mathcal{N} = \{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_i \text{ for } 1 \leq i \leq n\}$$

*is a  $\mathbf{k}$ -basis of  $\mathcal{A}$ .*

- (4) *The set  $\mathfrak{R}^\perp$  is a Gröbner basis of the ideal  $(\mathfrak{R}^\perp)$ .*
- (5) *The set  $(\mathfrak{R}^!)_0$  is a Gröbner basis, of the ideal  $((\mathfrak{R}^!)_0)$ .*
- (6) *The set of monomials*

$$\mathcal{N}^! = \{x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \mid 0 \leq \varepsilon_i \leq 1, \text{ for } 1 \leq i \leq n\}$$

*is a  $\mathbf{k}$ -basis of the Koszul dual algebra  $\mathcal{A}^!$ .*

- (7)  *$\mathcal{N}^!$  is a  $\mathbf{k}$ -basis of the semigroup algebra  $\mathbf{k}S^!$ , so  $S^!$  can be identified as a set with  $\mathcal{N}^!$ .*

**Definition 4.5.** [39], [40] A graded algebra  $A = \bigoplus_{i \geq 0} A_i$  is called a *Frobenius algebra of dimension  $n$* , (or a *Frobenius quantum space of dimension  $n$* ) if

- (a)  $\dim(A_n) = 1$ ,  $A_i = 0$ , for  $i > n$ .
- (b) For all  $n \geq j \geq 0$  the multiplicative map  $m : A_j \otimes A_{n-j} \rightarrow A_n$  is a perfect duality (nondegenerate pairing).

A Frobenius algebra  $A$  is called a *quantum Grassmann algebra* if in addition

- (c)  $\dim_{\mathbf{k}} A_i = \binom{n}{i}$ , for  $1 \leq i \leq n$ .

**Definition 4.6.** [15] In notation as above suppose the Koszul dual  $A^! = A^!(\mathfrak{R}_0)$  is Frobenius. The one dimensional component  $A_n^!$  is called *the socle of  $A^!$* . Let  $\mathcal{W} = \mathcal{W}(\xi_1, \xi_2, \dots, \xi_n)$  be the monomial which spans the socle. In this case the monoid  $S^!$  has top degree  $n$  and contains exactly one (nonzero) element  $W = W(x_1, \dots, x_n)$  of length  $n$ . Clearly, its pull-back  $\varphi^{-1}(W)$  in  $S$ , also contains exactly one element,  $W$ . Note that in this case  $W$  is the unique square-free element in  $S$  with length  $|W| = n$ . We call it *the principal monomial of  $S$*  (respectively *the principal monomial of  $\mathcal{A}$* ), see [15]. Suppose  $W$  can be presented as a product of exactly  $n$  generators,

$W = y_1 \cdots y_n, y_i \in X, y_i \neq y_j$  whenever  $i \neq j$ . So  $X = \{y_1, \dots, y_n\}$ . Fix the degree-lexicographic ordering  $<$  on  $\langle X \rangle$ , where  $y_1 < \dots < y_n$ . Denote  $\omega_0 = y_1 \cdots y_n$  (considered as a word in  $\langle X \rangle$ ), and look at the orbit  $\mathcal{O} = \mathcal{O}_{\mathcal{D}_n}(\omega_0)$ , in  $X^k$ . ( $\omega \in \mathcal{O}$  iff  $\omega = \omega_0 = W$  as elements of  $S$ ). By Definition 1.8,  $W = y_1 \cdots y_n$  is a *regular presentation* of  $W$  if  $\omega_0$  is the minimal element of  $\mathcal{O}$  with respect to the degree-lexicographic ordering  $<$ . The principal monomial  $W$  is *regular* if it has some regular presentation. In this case  $\mathcal{A}^!$  is said to have a *regular socle*.

The following can be extracted from [15, Theorem B] (bearing in mind that lri implies the cyclic conditions, see Remark 2.8).

**Facts 4.7.** [15] *Let  $(X, r)$  be a quantum binomial set with lri.  $S = S(X, r)$ ,  $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle / (\mathfrak{R}_0(r))$ , as usual. Then the following three conditions are equivalent.*

- (1)  *$S(X, r)$  is a regular quantum monoid, that is the Koszul dual  $A^!$  is Frobenius of dimension  $n$ , and has a regular socle.*
- (2)  *$S$  is a binomial skew polynomial monoid, with respect to some appropriate enumeration of  $X$ .*
- (3)  *$\mathcal{A}(\mathbf{k}, X, r)$  is a binomial skew polynomial ring, with respect to some appropriate enumeration of  $X$ .*
- (4)  *$(X, r)$  is a solution of YBE.*

The following result shows that a milder assumption:  $A^!(X, r)$  is Frobenius, and  $S$  is cancellative, ( $W$  is not necessarily regular), imply that  $(S, W)$  is a Garside monoid.

**Theorem 4.8.** *Let  $(X, r)$  be quantum binomial set with lri. Suppose the Koszul dual  $A^!(X, r)$  is Frobenius. Let  $W$  be the principal monomial in the monoid  $S = S(X, r)$ . Then*

- (1) *The set  $\sum_W$  of all left divisors of  $W$  in  $S$  consists exactly of all square-free elements  $u \in S$  of length  $|u| \leq n$  and coincides with the set of all right divisors of  $W$ .*
- (2) *Furthermore, if  $S$  is cancellative, then  $W$  is a Garside element, and  $(S, W)$  is a comprehensive Garside monoid.*

*Proof.* Note that  $W$  is square-free element of  $S$  with length  $|W| = n$ , see Definition 4.6. Then, clearly, each left (and right) divisor  $a$  of  $W$  has length  $\leq n$  and is square-free. Conversely, let  $a \in S$  be a square-free element. It follows from the Frobenius property that  $a$  has length  $\leq n$ , and if  $a$  has length  $n$  then  $a = W$ . Assume  $m = |a| < n$ . One has  $0 \neq a' = \varphi(a) \in \mathcal{A}_m^!$ , so by the Frobenius property, see Definition 4.5 (b), there exists a monomial  $b \in \mathcal{A}_{n-m}^!$  such that  $0 \neq ab \in \text{Span } W = \mathcal{A}_n^!$ . Clearly then  $ab = W$  holds in  $S$ , that is  $a$  is a left divisor of  $W$ . Analogously one shows that  $a$  is a right divisor of  $W$ . We have verified that each square-free element  $a \in S$  (of length  $\leq n$ ) is a left and a right divisor of  $W$ .

Assume furthermore that  $S$  is cancellative. Then condition (1. b) of Definition 1.6 is satisfied, so  $W$  is a Garside element and  $(S, W)$  is a pre-Garside monoid. We claim that  $(S, W)$  is a Garside monoid, that is condition (2) in Definition 1.6 holds.  $S$  is atomic with set of atoms  $X$ , see Remark 1.7. By hypothesis  $(X, r)$  is a quantum binomial set with lri so by Proposition 2.3 the left and right Ore conditions on generators are satisfied. Hence for arbitrary pair  $s, t \in X, s \neq t$ , there are uniquely determined  $x, y \in X$  such that  $sx = ty$  holds in  $S$ , the element

$\Delta_{s,t} = sx = ty$  is square-free and therefore is a left divisor of  $W$ . Clearly  $\Delta_{s,t}$  is the unique minimal element of the set

$$\{a \in \sum \mid s \preceq a, \text{ and } t \preceq a\}$$

with respect to the partial ordering  $\preceq$  on  $S$  induced by the left divisibility, ( $u \preceq v$  iff  $u$  is a left divisor of  $v$ ). It is straightforward that  $(S, W)$  is comprehensive, see Definition 1.6 (3).  $\square$

**Corollary 4.9.** *Suppose  $S = S(X, r)$  is a monoid of skew-polynomial type, with respect to the enumeration  $X = \{x_1, \dots, x_n\}$ . Then  $S$  is a regular Garside monoid with regular Garside element  $\Delta = x_1 x_2 \cdots x_n$ .*

*Proof.* By [10, Theorem 5.13]  $S$  is cancellative. Furthermore, the Koszul dual  $\mathcal{A}^!$  is Frobenius, see [15, Theorem 3.1], and  $W = x_1 \cdots x_n$  is a regular presentation of the principal monomial of  $S$ . It follows then by Theorem 4.8 that  $(S, W)$  is a Garside monoid, and clearly its Garside element  $W$  has regular presentation.  $\square$

**Proof of the Main Theorem.** Let  $(X, r)$  be a quantum binomial set with **lri**. For the conditions (1), (2), (3) of the theorem we have the following implications:

- (1)  $\implies$  (2) by Theorem 3.2
- (2)  $\iff$  (3) by [15, Theorem B], see also Facts 4.7
- (3)  $\implies$  (1) by Corollary 4.9

By Theorem 4.8,  $S$  is a comprehensive Garside monoid with Garside element  $\Delta$ . By [27], the monoid  $S(X, r)$  is embedded in its group of fraction, which is isomorphic to the group  $G = G(X, r)$ . Thus  $(G, \Delta)$  is a Garside group.

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